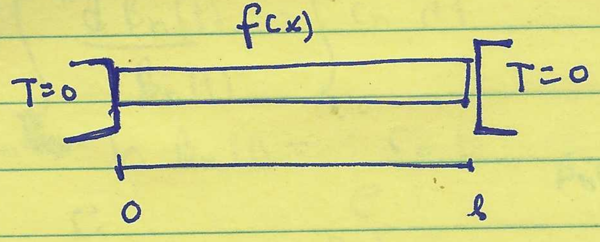
~~Barra a temperatura inicial con f(x) y que tiene~~ condiciones ~~de frontera~~

Barra finita de longitud l. Condiciones de frontera cero. Temperatura inicial cero.



$$T(x=0, t=0) = 0$$
$$T(x=l, t=0) = 0$$
$$T(x, t=0) = f(x)$$
$$x>0 \text{ y } x<l$$

$$\frac{\partial}{\partial t} T(x,t) = D \frac{\partial^2}{\partial x^2} T(x,t)$$

Se propone

$$T(x,t) = \sum_n T_n(x,t) = \sum_n A_n(x) B_n(t)$$

$$\frac{\partial}{\partial t} \sum_n A_n(x) B_n(t) = D \frac{\partial^2}{\partial x^2} \sum_n A_n(x) B_n(t)$$

$$\sum_n \left\{ \frac{\partial}{\partial t} A_n(x) B_n(t) \right\} = \sum_n \left\{ D \frac{\partial^2}{\partial x^2} A_n(x) B_n(t) \right\}$$

entonces se cumple

$$\frac{\partial}{\partial t} A_n(x) B_n(t) = D \frac{\partial^2}{\partial x^2} A_n(x) B_n(t)$$

$$\frac{1}{\cancel{A_n(x)} B_n(t)} \frac{\partial}{\partial t} \cancel{A_n(x)} B_n(t) = D \frac{1}{A_n(x) \cancel{B_n(t)}} \frac{\partial^2}{\partial x^2} A_n(x) \cancel{B_n(t)}$$

$$\frac{1}{B_n(t)} \frac{\partial}{\partial t} B_n(t) = D \frac{1}{A_n(x)} \frac{\partial^2}{\partial x^2} A_n(x) = + C_n$$

Sea

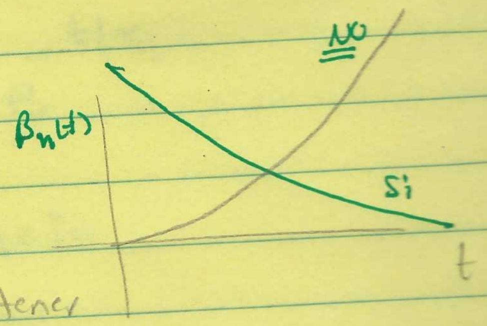
$$\frac{1}{B_n(t)}\frac{\partial}{\partial t}B_n(t) = C_n$$

$$\frac{\partial B_n(t)}{\partial t\, B_n(t)} = C_n\, \partial t$$

$$\int^{B_n(t)} \frac{d B_n(t)}{B_n(t)} = \int_{t_0}^{t} C_n\, dt$$

$$b_n\, \ln B_n(t) = C_n t$$

$$e \qquad e$$



$$B_n(t) = B_n e^{C_n t}$$

No es posible, así $C_n \to -C_n$ pa tener

$$\boxed{B_n(t) = B_n e^{-C_n t}}$$

Retomemos

$$\frac{\partial^2}{\partial x^2}A_n(x) = \frac{C_n}{D}A_n(x)$$

sujeto a

$$\left.\begin{aligned} A_n(x=0) &= 0 \\ A_n(x=\ell) &= 0 \end{aligned}\right\} t=0$$

Se propone

$$A_n(x) = a_n\, \mathrm{sen}\left(\frac{n\pi}{\ell}x\right)$$

así tenemos

$$-a_n\left(\frac{n\pi}{\ell}\right)^2 = -\frac{C_n}{D}a_n \Rightarrow C_n = D\left(\frac{n\pi}{\ell}\right)^2$$

así $\qquad B_n(t) = \cancel{B}\, b_n e^{-D\left(\frac{n\pi}{\ell}\right)^2 t}$

La solución es

$$T(x,t) = \sum_n a_n\, \mathrm{sen}\left(\frac{n\pi}{\ell}x\right) b_n e^{-D\left(\frac{n\pi}{\ell}\right)^2 t}$$

La solucion es

$$T(x,t) = \sum_{n=1}^{\infty} t_n \, \text{sen}\left(\frac{n\pi}{\ell} x\right) e^{-D\left(\frac{n\pi}{\ell}\right)^2 t}$$

Sea $\frac{1}{\tau_n} = D\left(\frac{n\pi}{\ell}\right)^2$

asi

$$T(x,t) = \sum_{n=1}^{\infty} t_n \, \text{sen}\left(\frac{n\pi}{\ell} x\right) e^{-t/\tau_n}$$

Para $t=0$, tenemos que $f(x) = T_0$

$$T_0 = \sum_{n=1}^{\infty} t_n \, \text{sen}\left(\frac{n\pi}{\ell} x\right)$$

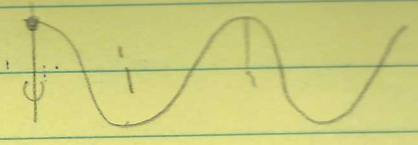
$$\int_0^{\ell} T_0 \, \text{sen}\left(\frac{m\pi}{\ell} x\right) dx = \sum_{n=1} t_n \int_0^{\ell} \text{sen}\left(\frac{n\pi}{\ell} x\right) \text{sen}\left(\frac{m\pi}{\ell} x\right) dx$$

$$T_0 \, I_1(m) = \sum_{n=1} t_n \, I_2(n,m)$$

$$I_1(m) = \int_0^{\ell} \text{sen}\left(\frac{m\pi}{\ell} x\right) dx$$

$$I_2(n,m) \int_0^{\ell} \text{sen}\left(\frac{n\pi}{\ell} x\right) \text{sen}\left(\frac{m\pi}{\ell} x\right) dx$$

$$I_1(n) = \int_0^l \operatorname{sen}\left(\frac{n\pi}{l}x\right) dx$$



$$= \frac{l}{n\pi}(-1)\cos \frac{n\pi}{l}x \Big|_0^l$$

$$= -\frac{l}{n\pi}\left[\cos(n\pi) - 1\right]$$

d sen = cos
d cos = -sen

Si n es par
$I_1(n) = 0$
Si n es impar
$$\boxed{I_1(n) = \frac{l}{n\pi}(+2)}$$

Ahora

$$I_2(n,m) = \int_0^l \operatorname{sen}\left(\overbrace{\frac{n\pi}{l}x}^{\theta_n}\right) \operatorname{sen}\left(\overbrace{\frac{m\pi}{l}x}^{\theta_m}\right) dx$$

$$= \int_0^l \left(\frac{e^{i\theta_n} - e^{-i\theta_n}}{2i}\right)\left(\frac{e^{i\theta_m} - e^{-i\theta_m}}{2i}\right) dx$$

$$= -\frac{1}{4}\int_0^l \left[e^{i(\theta_n+\theta_m)} - e^{i(\theta_n-\theta_m)} - e^{-i(\theta_n-\theta_m)} + e^{-i(\theta_n+\theta_m)}\right] dx$$

Si $\theta_n = \theta_m$

$$I_2(n, m=n) = -\frac{1}{4}\int_0^l \left[e^{i(\theta_n+\theta_n)} + e^{-i(\theta_n+\theta_n)} - 2\right] dx$$

$$= +\frac{1}{2}\int_0^l dx - \frac{1}{4}\,2\int_0^l \cos(2\theta_n)\, dx$$

$$= \frac{l}{2} - \frac{2}{4}\frac{1}{2\theta_n}\left[\operatorname{sen}(\cancel{2\theta_n}) - 0\right]$$

$$= \frac{l}{2}$$

Si $n \neq m$

$$I_2(n,m) = 0$$

así:

$$I_2 = \frac{l}{2}\,\delta_{n,m}$$

así tenemos

$$T_0 \frac{2l}{n\pi} = t_n \frac{l}{2}$$

$$\Rightarrow t_n = \frac{4}{n\pi} T_0$$

La solución es

$$T(x,t) = \sum_{n} \frac{4T_0}{n\pi} \operatorname{sen}\left(\frac{n\pi}{l}x\right) e^{-D\left(\frac{n\pi}{l}\right)^2 t}$$

$n = 1, 3, 5, 7$