Finally, consider a composite wall as before, but with contact resistances between the layers such that the flux of heat between the surfaces of consecutive layers is H times the temperature difference between these surfaces (cf. 1.9 (20)). Here $1/H$ may be regarded as the thermal resistance of the contact, and the total thermal resistance of the composite wall will be the sum of the thermal resistances of the separate layers plus the sum of the thermal resistances of the contacts between them.

If the conductivity K is a function of the temperature, the differential equation is

$$\frac{d}{dx}\left(K\frac{dv}{dx}\right) = 0.$$

Thus the relation $$-K\frac{dv}{dx} = f, \quad \text{constant,}$$

still holds. Integrating between the surface temperatures v_1 and v_2 of a slab of thickness l we have

$$-\int_{v_1}^{v_2} K\,dv = lf,$$

and thus $$f = \frac{(v_1-v_2)K_{av}}{l}, \tag{4}$$

where $$K_{av} = \frac{1}{v_2-v_1}\int_{v_1}^{v_2} K\,dv \tag{5}$$

is the average conductivity over the temperature range in the slab. Thus, if conductivity is a function of temperature, the previous results hold good with K_{av} in place of K.

3.3. The region $0 < x < l$. Ends kept at zero temperature. Initial temperature $f(x)$

The differential equation to be solved is

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad 0 < x < l, \tag{1}$$

with $$v = 0, \quad \text{when } x = 0 \text{ and } x = l, \tag{2}$$

and $$v = f(x), \quad \text{when } t = 0. \tag{3}$$

If the initial distribution were

$$v = A_n \sin\frac{n\pi x}{l},$$

it is clear that $$v = A_n \sin\frac{n\pi x}{l}\, e^{-\kappa n^2\pi^2 t/l^2}$$

would satisfy all the conditions (1), (2), (3) of the problem.

Let us suppose that the initial temperature, $f(x)$, is a bounded function satisfying Dirichlet's conditions† (*F.S.*, § 93) in the interval $(0, l)$ so that it can be expanded in the sine series

$$\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l},$$

where
$$a_n = \frac{2}{l}\int_0^l f(x')\sin\frac{n\pi x'}{l}\,dx'. \tag{4}$$

Now consider the function v defined by the infinite series

$$v = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} e^{-\kappa n^2\pi^2 t/l^2}. \tag{5}$$

This series, owing to the presence of the convergency factor $\exp[-(\kappa n^2\pi^2 t/l^2)]$, is uniformly convergent‡ for any interval of x, when $t > 0$; and, regarded as a function of t, it is uniformly convergent when $t \geqslant t_0 > 0$, t_0 being any positive number.

Thus the function v, defined by the series (5), is a continuous function of x, and a continuous function of t, in these intervals.§

It is easy to show that the series obtained by term-by-term differentiation of (5) with respect to x and t are also uniformly convergent in these intervals of x and t respectively. Thus they are equal to the differential coefficients of the function v.

Hence
$$\frac{\partial v}{\partial t} = -\sum_{1}^{\infty} \frac{\kappa n^2\pi^2}{l^2} a_n \sin\frac{n\pi x}{l} e^{-\kappa n^2\pi^2 t/l^2}$$

and
$$\kappa\frac{\partial^2 v}{\partial x^2} = -\sum_{1}^{\infty} \frac{\kappa n^2\pi^2}{l^2} a_n \sin\frac{n\pi x}{l} e^{-\kappa n^2\pi^2 t/l^2},$$

when $t > 0$, and $0 < x < l$.

† This restriction is removed in *C.H.* § 31 where it is shown that the results below hold if $f(x)$ is bounded and integrable in $0 \leqslant x \leqslant l$.

‡ Since $f(x)$ is bounded there is a positive number M such that $|f(x)| < M$ in $0 < x < l$. It follows that $|a_n| < 2M$ for all values of n. Therefore

$$\left|a_n \sin\frac{n\pi x}{l} e^{-\kappa n^2\pi^2 t/l^2}\right| < 2Me^{-\kappa n^2\pi^2 t_0/l^2}, \quad \text{where} \quad t \geqslant t_0 > 0.$$

Now the series
$$\sum_{n=1}^{\infty} e^{-\kappa n^2\pi^2 t_0/l^2}$$

is convergent and its terms are independent of both x and t, and the results follow.

§ Regarded as a function of the two variables x, t, it is a continuous function of (x, t) in the regions $0 \leqslant x \leqslant l$, $t \geqslant t_0 > 0$. (Cf. *F.S.*, § 37.)

Thus the equation $$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}$$

is satisfied at all points of the rod, when $t > 0$, by the function defined by (5).

We have now to see whether this function also satisfies the boundary and initial conditions.

Since the series is uniformly convergent with respect to x in the interval $0 \leqslant x \leqslant l$, when $t > 0$, it represents a continuous function of x in this interval.

Thus

$$\lim_{x\to 0} v = \text{the value of the sum of the series when } x = 0$$
$$= 0,$$

and $$\lim_{x\to l} v = \text{the value of the sum of the series when } x = l$$
$$= 0.$$

Hence the *boundary conditions* are satisfied.

With regard to the *initial conditions*, we may use the extension of Abel's theorem contained in *F.S.*, § 73, I.

We have assumed that $f(x)$ is bounded and satisfies Dirichlet's conditions in $(0, l)$.

Therefore the sine series for $f(x)$,

$$a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{2\pi x}{l} + \dots,$$

converges, and its sum is $f(x)$ at every point between 0 and l where $f(x)$ is continuous, and $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at all other points.† (Cf. *F.S.*, § 98.)

It follows from the extension of Abel's theorem referred to above that when v is defined by (5), we have

$$\lim_{t\to 0} v = \lim_{t\to 0} \sum_1^\infty a_n \sin\frac{n\pi x}{l} e^{-\kappa n^2\pi^2 t/l^2}$$
$$= f(x) \quad \text{at a point of continuity}$$
$$= \tfrac{1}{2}\{f(x+0)+f(x-0)\} \quad \text{at all other points.}$$

Thus we have shown that if the initial temperature satisfies Dirichlet's

† If $f(x)$ is bounded and satisfies Dirichlet's conditions, it follows from *F.S.*, § 93, that it can only have ordinary discontinuities.

conditions, and is continuous from $x = 0$ to $x = l$, while $f(0) = f(l) = 0$, the function defined by (5)† satisfies all the conditions of the problem.

If the initial temperature has discontinuities, the function defined by (5) at these points tends to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ as $t \to 0$. If t is taken small enough, v will bridge the gap from $f(x-0)$ to $f(x+0)$, and the temperature curve will pass close the point $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

It must be remembered that the physical problem, as we have stated it for discontinuity, either at the ends of the rod or in the rod itself, is an ideal one. In nature there cannot be a discontinuity in the temperature in the rod initially. In the physical problem we must assume that a sudden change of temperature takes place at the instant from which our observations are measured, in the immediate neighbourhood of the point of discontinuity or the ends, if they are points of discontinuity. The gap in the temperature is thus smoothed over. The solution of the mathematical problem we have obtained satisfies these conditions, and it may be taken as representing the physical problem in this modified aspect.

The following special cases of (5) are of interest:‡

(i) *Constant initial temperature* $f(x) = V_0$, *constant.*

$$v = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}e^{-\kappa(2n+1)^2\pi^2t/l^2}\sin\frac{(2n+1)\pi x}{l}. \tag{6}$$

(ii) *A linear initial distribution* $f(x) = kx$.

$$v = \frac{2lk}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{-\kappa n^2\pi^2t/l^2}\sin\frac{n\pi x}{l}. \tag{7}$$

In general, it is a little more satisfactory to set out results for the symmetrical case of the slab $-l < x < l$ so that direct comparison with similar results for the sphere and cylinder is possible. Also, it is invariably found that series such as (6) and (7) converge slowly for small values of $\kappa t/l^2$, say $\kappa t/l^2 < 0{\cdot}01$, but it will appear later (cf. § 12.5) that alternative series involving error functions or their integrals are rapidly convergent for such values. For convenience these alternative series will be given here, cf. (9), (11), their derivation will be discussed in § 12.5. All the results given below hold, of course, also for the slab $0 < x < l$, with no flow of heat at $x = 0$, and $x = l$ at zero temperature.

† This can be written as

$$v = \frac{2}{l}\int_0^l f(x')\sum_1^{\infty}\left(\sin\frac{n\pi x'}{l}\sin\frac{n\pi x}{l}e^{-\kappa n^2\pi^2t/l^2}\right)dx',$$

since the series under the integral is uniformly convergent. (*F.S.*, § 70.)

‡ Series such as (6) can also be expressed in terms of theta functions, cf. Whittaker and Watson, *Modern Analysis* (Cambridge, edn. 3, 1920) Chap. XXI.

(iii) *The slab $-l < x < l$ with constant initial temperature V_0.*

Changing the origin in (6) to the mid-point of the slab and replacing $\frac{1}{2}l$ by l gives

$$v = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}e^{-\kappa(2n+1)^2\pi^2t/4l^2}\cos\frac{(2n+1)\pi x}{2l} \tag{8}$$

$$= V_0 - V_0\sum_{n=0}^{\infty}(-1)^n\left\{\operatorname{erfc}\frac{(2n+1)l-x}{2(\kappa t)^{\frac{1}{2}}}+\operatorname{erfc}\frac{(2n+1)l+x}{2(\kappa t)^{\frac{1}{2}}}\right\}. \tag{9}$$

Some numerical results for this problem are given in Figs. 10 (*a*) and 11.

The average temperature v_{av} in the slab at time t is

$$v_{av} = \frac{8V_0}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}e^{-\kappa(2n+1)^2\pi^2t/4l^2} \tag{10}$$

$$= V_0 - 2V_0\left(\frac{\kappa t}{l^2}\right)^{\frac{1}{2}}\left\{\pi^{-\frac{1}{2}}+2\sum_{n=1}^{\infty}(-1)^n\operatorname{ierfc}\frac{nl}{(\kappa t)^{\frac{1}{2}}}\right\}. \tag{11}$$

The quantity of heat per unit area of the slab at time t is just $2l\rho c v_{av}$. Measurements of this quantity are frequently used to determine diffusivities or diffusion coefficients.†

The flux of heat f at the surface is

$$f = -K\left[\frac{\partial v}{\partial x}\right]_{x=l} = \frac{2KV_0}{l}\sum_{n=0}^{\infty}e^{-\kappa(2n+1)^2\pi^2t/4l^2} \tag{12}$$

$$= \frac{KV_0}{(\pi\kappa t)^{\frac{1}{2}}}\left\{1+2\sum_{n=1}^{\infty}(-1)^n e^{-n^2l^2/\kappa t}\right\}. \tag{13}$$

Ingersoll and Koepp‡ have used this solution for the determination of κ for earth materials; also Frazier§ has used it for metal rods by observing the difference in temperature between the points $x = a$ and $x = b$ of the rod. He chooses a and b so that

$$\cos(3\pi a/2l) = \cos(3\pi b/2l).$$

In this case the second term of the series derived from (8) for the temperature difference vanishes, and the third term which has $\exp[-25\kappa\pi^2t/4l^2]$ as a factor disappears very rapidly.

(iv) *The region $-l < x < l$ with initial temperature $V_0(l-|x|)/l$ and zero surface temperature.*

$$v = \frac{8V_0}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{2l}e^{-\kappa(2n+1)^2\pi^2t/4l^2} \tag{14}$$

$$= \frac{V_0(l-|x|)}{l}-\frac{2V_0(\kappa t)^{\frac{1}{2}}}{l}\sum_{n=0}^{\infty}(-1)^n\left\{\operatorname{ierfc}\frac{2nl+|x|}{2(\kappa t)^{\frac{1}{2}}}-\operatorname{ierfc}\frac{(2n+2)l-|x|}{2(\kappa t)^{\frac{1}{2}}}\right\}. \tag{15}$$

† Anderson and Saddington, *J. Chem. Soc.* (1949) 381–6.
‡ *Phys. Rev.* (2) **24** (1924) 92.
§ *Phys. Rev.* (2) **39** (1932) 515.

(v) *The region* $-l < x < l$ *with initial temperature* $V_0(l^2-x^2)/l^2$ *and zero surface temperature.*

$$v = \frac{32V_0}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}e^{-\kappa(2n+1)^2\pi^2t/4l^2}\cos\frac{(2n+1)\pi x}{2l} \tag{16}$$

$$= \frac{V_0(l^2-x^2)}{l^2}-\frac{2\kappa tV_0}{l^2}+\frac{8\kappa tV_0}{l^2}\sum_{n=0}^{\infty}(-1)^n\left\{\mathrm{i^2erfc}\frac{(2n+1)l-x}{2(\kappa t)^{\frac{1}{2}}}+\mathrm{i^2erfc}\frac{(2n+1)l+x}{2(\kappa t)^{\frac{1}{2}}}\right\}. \tag{17}$$

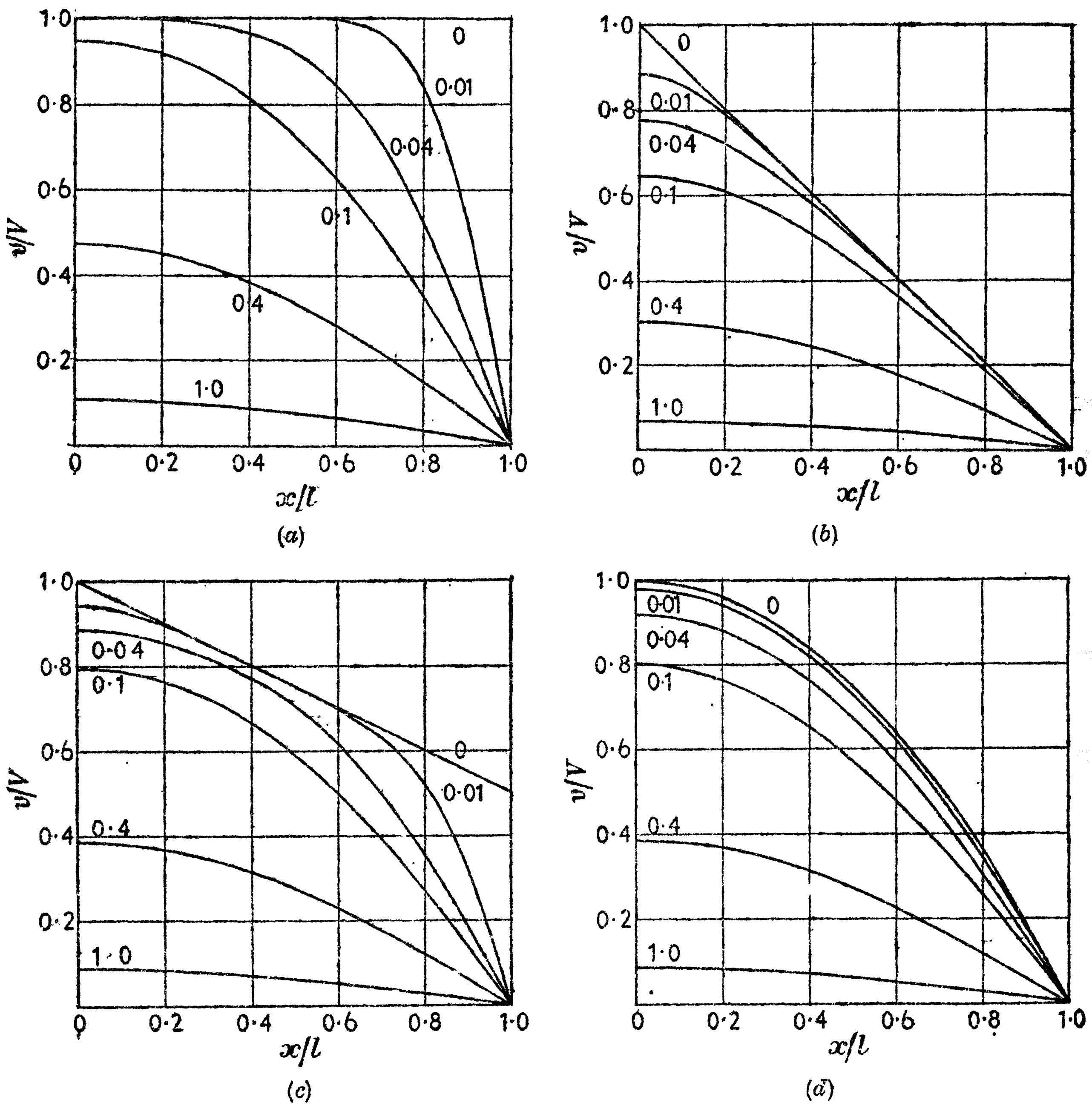

FIG. 10. Temperatures in the slab $0 < x < l$ with no flow at $x = 0$, zero temperature at $x = l$, and various initial distributions of temperature. The numbers on the curves are the values of $\kappa t/l^2$. (*a*) Constant initial temperature; (*b*) linear initial temperature, § 3.3 (iv); (*c*) 'linear + constant' initial temperature; (*d*) parabolic initial temperature, § 3.3 (v)

(vi) *The region $-l < x < l$ with initial temperature $V_0 \cos(\pi x/2l)$ and zero surface temperature.*

$$v = V_0 \cos\frac{\pi x}{2l} e^{-\kappa\pi^2 t/4l^2}. \tag{18}$$

These results are interesting since they give a qualitative idea of the way in which heat is extracted from a slab with a given initial distribution of temperature. It appears from (5) that the higher harmonics in the Fourier series for $f(x)$ disappear first, leaving the fundamental whose amplitude diminishes exponentially. This is, in effect, restated in (18). In Fig. 10 (*a*)–(*d*) the decay of temperature for four different initial distributions of temperature is shown, viz. constant, linear, 'linear+constant', and parabolic. It appears that heat is removed in such a way as to make the distribution approximate to a cosine: for a constant distribution heat is taken first from near the surface; for a linear distribution from near the centre; for a 'linear+constant' distribution from both centre and surface.

3.4. The region $0 < x < l$. Initial temperature $f(x)$. The ends at constant temperature or insulated

In the case in which the ends are kept at constant temperatures v_1 and v_2 we have the equations

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} \quad (0 < x < l),$$

$$v = v_1, \quad \text{when } x = 0,$$

$$v = v_2, \quad \text{when } x = l,$$

and

$$v = f(x), \quad \text{when } t = 0.$$

As in § 1.14, we reduce this to a case of steady temperature, and a case where the ends are kept at zero temperature.

Put

$$v = u+w,$$

where u and w satisfy the following equations:

$$\frac{d^2 u}{dx^2} = 0 \quad (0 < x < l),$$

$$u = v_1, \quad \text{when } x = 0,$$

$$u = v_2, \quad \text{when } x = l,$$

and

$$\frac{\partial w}{\partial t} = \kappa \frac{\partial^2 w}{\partial x^2} \quad (0 < x < l),$$

$$w = 0, \quad \text{when } x = 0 \text{ and } x = l,$$

$$w = f(x) - u, \quad \text{when } t = 0.$$

We find at once that

$$u = v_1 + (v_2 - v_1)x/l,$$